Anti-phase locking in a two-dimensional Josephson junction array

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Abstract

We consider theoretically phase locking in a simple two-dimensional Josephson junction array consisting of two loops coupled via a joint line transverse to the bias current. Ring inductances are supposed to be small, and special emphasis is taken on the influence of external flux. It is shown, that in the stable oscillation regime both cells oscillate with a phase shift equal to π (i.e. anti-phase). This result may explain the low radiation output obtained so far in two-dimensional Josephson junction arrays experimentally.

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Josephson Junction arrays are currently under consideration as tunable microwave radiation sources.[1,2]. After clarifying the basic principles of one-dimensional (1D) arrays[3–7], there is a growing interest in two-dimensional (2D) arrays since the pioneering work by Benz and Booi[8–11]. According to basic estimates[2] radiation output generated by this type of arrays is expected to be much larger than that from 1D arrays. However, experimental results point just in the opposite direction. While linear arrays delivered output powers up to 50μW, the maximum output power reported for two-dimensional arrays is around 100nW,[12–14] and usually much smaller. While the general estimates referred to above are surely true, this should be caused by the fact, that actually very few junctions are locked in-phase.

There can be several reasons responsible for poor radiation output. Besides comparably small critical currents as in[22] or technological problems this can as well result from the fact, that the basic mechanisms of phase locking in 2D arrays, despite some interesting results on several aspects[2,15,13,16], are not yet sufficiently understood. It is well known though, that there is no phase locking in unshunted 2D arrays in the absence of external flux. A theoretical study of the influence of flux with a "master-slave-mechanism" by Filatrella and Wiesenfeld[15] led to the conclusion that external flux can indeed lead to a certain phase locking; however the definite value of the phase difference could not be determined by their method, and stability was not considered at all.

Here, we start with a very simple model 2D array, consisting of two loops coupled via a line transverse to the bias current (Fig. 1). Despite its simplicity this model is difficult enough to show the essential features of larger arrays: It is truly two-dimensional with a possible external flux entering the loops and a inductance in the transverse line, as it is typical in the nowadays favored hybrid arrays. Our propositions are as follows: (i) Both junctions are considered to be identical. (ii) Self-inductance is taken into account while mutual inductances are neglected. (iii) Junctions are overdamped with $\beta \approx 0$. (iv) There is no external load. (v) Instead of working within the framework of the widely used first harmonic approximation we exploit a phase slip technique which has proven successful in 1D arrays before[3,6,7]. Its applicabiliy crucially depends on the proposition that the normalized ring inductance

$$l = 2\pi I_C L/\Phi_0 \tag{1}$$

(I_C critical current of the junctions, L ring inductance, Φ_0 flux quantum) is sufficiantly small ($l \ll 1$).

Josephson junctions are described by the RSJ (Resistively Shunted Junction) equations for the Josephson phases ϕ_{jk},

$$\dot{\phi}_{jk} + \sin\phi_{jk} = i_{jk} \quad (\{j,k\} = \{1,2\}), \tag{2}$$

where the dot denotes differentiation w.r.t. the normalized time variable

$$s = \frac{2\mathrm{e}}{\hbar} R_N I_C t \tag{3}$$

(R_N: junction normal resistance; all currents are normalized to I_C). Normalizing the external magnetic flux Φ according to

$$\varphi = 2\pi\Phi/\Phi_0 \tag{4}$$

we have to respect two flux quantization conditions,

$$\phi_{12} - \phi_{11} - \varphi - l\bar{i} = 0, \tag{5}$$

$$\phi_{22} - \phi_{21} - \varphi + l\bar{i} = 0. \tag{6}$$

In the following the transverse current playing a crucial role in the coupling will be denoted by $\bar{i}$ (cf. Fig. 1). In strong coupling problems of this type it has proven useful to introduce sum and difference variables according to[17,18]

$$\Sigma_k = \frac{1}{2}(\phi_{k2} + \phi_{k1}), \tag{7}$$

$$\Delta_k = \frac{1}{2}(\phi_{k2} - \phi_{k1}). \tag{8}$$

In addition, we introduce the circular currents

$$i_k^\circ = (i_{k2} - i_{k1})/2. \tag{9}$$

With the help of these variables the problem can be reformulated as

$$\dot{\Sigma}_k + \sin \Sigma_k \cos \Delta_k = i_0, \tag{10}$$

$$\dot{\Delta}_k + \sin \Delta_k \cos \Sigma_k = i_k^\circ, \tag{11}$$

$$\Delta_1 + \Delta_2 - \varphi = 0, \tag{12}$$

$$\Delta_1 - \Delta_2 - l(i_2^\circ - i_1^\circ) = 0. \tag{13}$$

This indicates, that the voltage sums of both loops are driven by the bias current $i_0 > 1$, while the circular currents drive voltage differences. Further, Eq. (12) is the flux quantization for the whole array, while Eq. 13 shows that differences in the circular currents spread the flux differences of the loops. The transverse current $\bar{i}$ can be obtained from

$$\bar{i} = i_2^\circ - i_1^\circ = \frac{1}{l}(\Delta_1 - \Delta_2). \tag{14}$$

According to Eq. (13) it is just the combination $l\bar{i}$ which causes the coupling between the cells.

The system (10)-(13) is treated perturbatively assuming the ring inductance l to be sufficiently small. To lowest order, the flux quantization conditions gives (the second index indicating the order of evaluation)

$$\Delta_{k,0} = \varphi/2, \tag{15}$$

i.e., junctions within both loops oscillate exactly in-phase. The Josephson oscillations itself can be evaluated from (10) as

$$\Sigma_{i,0} = \frac{\pi}{2} + 2\arctan\frac{\zeta_0}{i_0 + \cos(\varphi/2)}\tan\left(\frac{\zeta_0 s - \delta_i}{2}\right), \tag{16}$$

where we introduced the flux-dependent autonomous oscillation frequency

$$\zeta_0 = \sqrt{(i_0^2 - \cos^2(\varphi/2))}. \tag{17}$$

This already completes the lowest order solution for our problem; Eqs. (11) are not required for evaluating the Josephson phases within this order, but determine the circular currents

$$i^{\circ}_{k,0} = \sin(\varphi/2)\cos\Sigma_{k,0} \tag{18}$$

with

$$\cos\Sigma_{k,0} = -\frac{\zeta_0 \sin(\zeta_0 s - \delta_k)}{i_0 + \cos(\varphi/2)\cos(\zeta_0 s - \delta_k)}. \tag{19}$$

To summarize, in lowest order the junctions within each cell oscillate in phase independently of the value of the external flux, while the relative oscillation phase between the cells remains undetermined.

Changing to the next order l^1 we start again from the Josephson phase differences (12) and (13), inserting the lowest order result (18) on the r.h.s. of Eq. (13). From the two algebraic equations arising the correction terms $\Delta_{k,1}$ can be easily evaluated, and the Josephson phase differences of the two loops up to the first order in l are given by

$$\Delta_1 = \frac{\varphi}{2} + \frac{l}{2}\sin(\varphi/2)(\cos\Sigma_{2,0} - \cos\Sigma_{1,0}), \tag{20}$$

$$\Delta_2 = \frac{\varphi}{2} - \frac{l}{2}\sin(\varphi/2)(\cos\Sigma_{2,0} - \cos\Sigma_{1,0}). \tag{21}$$

From this result, one can read off the transverse current

$$\bar{i} = \sin(\varphi/2)\left(\cos\Sigma_{2,0} - \Sigma_{1,0}\right) \tag{22}$$

with the basic harmonic

$$\bar{i} = \frac{4\zeta_0 \sin(\varphi/2)}{i_0 + \zeta_0}\cos\left(\zeta_0 s - \frac{\delta_1 + \delta_2}{2}\right)\sin\left(\frac{\delta_2 - \delta_1}{2}\right). \tag{23}$$

We point out, that although $\bar{i}$ is proportional to $1/l$ this factor cancels out because of $\Delta_1 - \Delta_2$ being proportional to l itself. Accordingly, the amplitude of the transverse current is the same independently of the inductance l.

The most remarkable property of this type of "internal shunt current" is its vanishing for $\varphi = 0$ and growing with the external flux φ. One should notice, that this behavior is just opposite to that of an external shunt current, which usually turns out to be proportional to $\cos(\varphi/2)$. The absence of any transverse rf current for $\varphi = 0$ is however obvious: In this case the array is completely symmetric.

For evaluating the Josephson phase sums of the cells we exploit the method of "slowly varying phase" which has proven useful in the study of phase locking in one-dimensional arrays before[6,7,4]. According to this method corrections are put into the phases δ_k,

$$\delta_k = \delta_k(s), \tag{24}$$

which are supposed to change adiabatically only (in comparison to the rf Josephson oscillations) in time. In addition, we will allow for the possibility that the joint oscillation frequency ζ be (slightly) different from the autonomous frequency ζ_0. With these assumptions the voltage sums can be written as

$$\dot{\Sigma}_k = \frac{\zeta_0(\zeta - \dot{\delta}_k)}{i_0 + \cos(\varphi/2)\cos(\zeta s - \delta_k)}. \tag{25}$$

Inserting (25) into (10) and neglecting higher orders in l after some algebra we arrive at

$$\zeta_0(\zeta - \zeta_0 - \dot{\delta}_1) = (l/4)\bar{i}(s)\sin\varphi + (l/2)i_0\bar{i}(s)\sin(\varphi/2)\cos(\zeta s - \delta_1), \tag{26}$$

$$\zeta_0(\zeta - \zeta_0 - \dot{\delta}_2) = -(l/4)\bar{i}(s)\sin\varphi - (l/2)i_0\bar{i}(s)\sin(\varphi/2)\cos(\zeta s - \delta_2). \tag{27}$$

Here, all the interaction terms proportional to l arising on the l.h.s. of Eq. (10) were transferred to the r.h.s. In this way, the combination $l\bar{i}$ plays a similar role as a synchronizing alternating external or shunt current[3,19,20].

To proceed, we average over one oscillation period, considering δ_k as roughly constant over this time interval. It can be shown, that only the lowest harmonic (23) of $\bar{i}$ contributes. Evaluation of the mean values results in the evolution equations

$$\zeta_0(\zeta - \zeta_0 - < \dot{\delta}_1 >) = l\frac{\zeta_0 i_0}{2(i_0 + \zeta_0)}\sin^2(\varphi/2)\sin(< \delta_2 > - < \delta_1 >), \tag{28}$$

$$\zeta_0(\zeta - \zeta_0 - < \dot{\delta}_2 >) = -l\frac{\zeta_0 i_0}{2(i_0 + \zeta_0)}\sin^2(\varphi/2)\sin(< \delta_2 > - < \delta_1 >), \tag{29}$$

where $< \delta_k >$ denotes the one-period average over δ_k. Subtraction gives the reduced equation for the phase difference $\delta = \delta_1 - \delta_2$,

$$< \dot{\delta} > = l\frac{i_0}{i_0 + \zeta_0}\sin^2(\varphi/2)\sin < \delta >, \tag{30}$$

having formally the same structure as the RSJ equation describing an unbiased autonomous junction. It admits two phase locking solutions,

$$< \delta^{\rm pl} >= 0 \quad \text{and} \quad < \delta^{\rm pl} >= \pi, \tag{31}$$

describing in-phase ($< \delta^{\rm pl} >= 0$) or anti-phase ($< \delta^{\rm pl} >= \pi$) oscillations of the cells. Investigation of the stability leads to the Liapunov coefficient

$$\lambda = l \frac{\dot{i}_0}{\dot{i}_0 + \zeta_0} \sin^2(\varphi/2) \cos < \delta^{\rm pl} > . \tag{32}$$

As a result, only anti-phase oscillations are stable against small perturbations. By substituting (31) into (28) one easily recovers that the oscillation frequency remains equal to that of an autonomous junction, i.e.

$$\zeta = \zeta_0. \tag{33}$$

To summarize, the following picture arises: From earlier results[17,21] we know, that the two junctions within each strongly coupled cell are generally (except for $\varphi \approx \pi$) aligned in-phase. In addition, according to (31) both junctions from cell one oscillate anti-phase relative to those from cell 2. Synchronization of the cells in this state is provided by the alternating current (22), flowing through the joint transverse connection. It is obvious, that such a state will be non-radiating. In addition, our findings justify earlier results on missing phase locking in the absence of external flux, which within our framework can be explained by the marginal stability observed in (32) for $\varphi = 0$.

All results described in this paper are in complete agreement with corresponding numerical simulations performed in parallel. These simulations show, that the observed anti-phase locking is not bounded to the case of small inductances treated analytically here, but is a general feature of this type of array. If this remains true for larger arrays, which is under investigation now, this might well explain the low radiation output obtained with two-dimensional arrays up to now. In addition, investigations are on the way on the interplay with an external shunt current.

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FIGURES

FIG. 1. The two-dimensional Josephson junction circuit under investigation.

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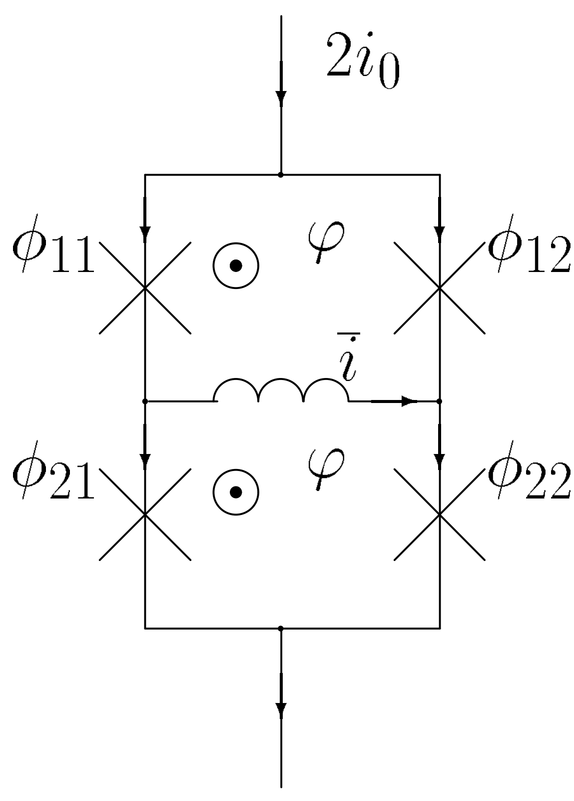

$2i_0$
ϕ_{11}
φ
ϕ_{12}
$\bar{i}$
ϕ_{21}
φ
ϕ_{22}